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ANY INFINITE-DIMENSIONAL FRÉCHET SPACE HOMEOMORPHIC WITH ITS COUNTABLE PRODUCT IS TOPOLOGICALLY A HILBERT SPACE

BY

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ABSTRACT. In this paper we will prove that any infinite-dimensional Fréchet space homeomorphic with its own countable product is topologically a Hilbert space. This will be done in two parts. First we will prove the result for infinite-dimensional Banach spaces, and then we will show that the result for Fréchet spaces follows as a corollary.

1. Introduction. Let F be a Fréchet space (complete locally convex metric topological vector space) such that F is homeomorphic with ($\cong$) its own countable product (F^{ω}). In the following we will show that such a Fréchet space is homeomorphic with a Hilbert space of appropriate weight.

In an addendum to [13], Toruńczyk claims a proof of the same result. The two proofs are independent and use techniques which are completely different.

2. Preliminaries. Let Λ be a set of cardinality $\aleph$. The space $l_p(\aleph)$ for fixed $p \geq 1$ is defined to be the set of all real functions $r = \{r_\lambda\}$ defined on the set Λ with at most a countable number of nonzero elements and with $\Sigma_\lambda |r_\lambda|^p < \infty$. The norm on $l_p(\aleph)$ is $\|r\| = \{\Sigma_\lambda |r_\lambda|^p\}^{1/p}$. When $p = 2$, this is a Hilbert space of weight $\aleph$.

In [1] Bessaga has proven the following theorem:

Theorem 1. *If F is a Fréchet space then $l_1(wF) \cong l_1(wF) \times F$ where wF is the cardinal equal to the weight of F.*

Proof. See 8.1, 3.2, 8.4, and 8.5 in [1]. □

We will use the existence of such a homeomorphism for a Banach space to show $B \cong B^{\omega}$ implies $B \cong l_1(wB)$.

We will now prove an imbedding theorem for Banach spaces.

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Lemma 2. *Let B be an infinite-dimensional Banach space. Then there is a closed imbedding of $l_1(wB)$ into B^ω.*

Proof. Fix $1/n$ with n a positive integer, and let $\{U'_\alpha\}_{\alpha\in A}$ be the collection of all $1/n$ balls of $l_1(wB)$. Find a locally finite refinement $\{U_\beta\}_{\beta\in \mathrm{B}}$ and let $\{\psi_\beta\}_{\beta\in\mathrm{B}}$ be a partition of unity subordinate to this cover. Applying a theorem of Michael (Lemma 2.1(c) and (e) of [11]) we may obtain a locally finite refinement $\{G^n_{i\beta}\}_{\beta\in\mathrm{B}^n_i}$, $i = 1, 2, \cdots$, such that $G_{i\beta}\cap G_{i\gamma} = \emptyset$ if $\beta\neq\gamma$. Do this for each positive integer. Next, by Lemma 1.2 of [4], pick a collection of disjoint open sets in the unit sphere of B having cardinality the weight of B, and pick one point x_β from each of the open sets.

Define

$$g\colon l_1(wB) \to \prod_{n=1}^{\infty}\left(\prod_{i=1}^{\infty} B_i\right)_n,$$
$$\{r_\beta\} \mapsto \left\{\sum_{i\beta\in\mathrm{B}^n_i} \psi^n_{i\beta}(r_{i\beta})x_{i\beta}\right\}.$$

Note that there is at most one nonzero coordinate in each B_i for each n since a point may lie in at most one subset of a disjoint collection.

Now, g is clearly continuous, and g is one-to-one since no point is within $1/n$ of any other point for all n. To see g^{-1} is continuous, observe that given $\epsilon>0$ and a sequence $\{y^j\}$ converging to y in the image of g, we may pick n such that $2/n<\epsilon$. Then pick a k such that y has a nonzero coordinate in $B_{k,n}$ where $B_{k,n}$ is the kth copy of B in the nth product $(\Pi_{i=1}^\infty B_i)_n$. Let $x_{k\beta}$ be a point with the nonzero real multiple. Then since convergence in a product is equivalent to coordinate-wise convergence, pick J such that $j>J$ implies the $x_{k\beta}$ multiple in $B_{k,n}$ is nonzero for y^j. It is here that we are using the fact that $\{x_\beta\}$ is a discrete set. Then $g^{-1}(y^j)$ and $g^{-1}(y)$ are contained in a $1/n$ ball in $l_1(wB)$ for all $j>J$. Therefore, the distance between $g^{-1}(y^j)$ and $g^{-1}(y)$ is less than ϵ for all $j>J$.

This gives an imbedding of $l_1(wB)$ as a G_δ set in B^ω. Now, given a Banach space B, by the Hahn-Banach theorem $B\cong \mathbf{R}\times N$ where $\mathbf{R}$ is a copy of the reals. Therefore,

$$B^\omega \cong (\mathbf{R}\times N)^\omega \cong \mathbf{R}^\omega\times N^\omega \cong \mathbf{R}^\omega\times B^\omega.$$

Using this fact, we may imbed this G_δ as a closed subset of B^ω. (See Kuratowski [9, pp. 229 and 430].) Thus we obtain a closed imbedding. □

Remark 1. The proof of Lemma 2 may be adapted to any open cone, M, which is a topological vector space. All we need to do is pick a discrete set $\{x_\beta\}$ in M^ω which is radially independent and for which the cardinality of $\{x_\beta\}$ equals the weight of M. See Lemma 1.2 of [4] to see that we may pick a discrete set $\{x_\beta\}$ chosen from disjoint open sets. In the nonseparable case, using a Zorn's lemma argument together with the fact that the topology of a ray is second countable, we may then produce a radially independent set. The separable case uses the fact that any metric topological vector space has a real factor. See Henderson's paper [5] for theorems on spaces being open cones.

Given $\{(B_i, \| \|_i) | i = 1, 2, \cdots\}$, a collection of Banach spaces, and given $l_1 = l_1(\aleph_0)$, we will define $\Sigma_{l_1} B_i$ to be the set of sequences $\{x_i\}$, $x_i \in B_i$, such that $\{\|x_i\|_i\} \in l_1$. If $(B_i, \| \|_i)$ is the same pair for each i, we will just write $\Sigma_{l_1} B$. It is easy to show that $\Sigma_{l_1} B$ is a Banach space.

Lemma 3. *Given an infinite-dimensional Banach space B, there is a homeomorphism*

$$h: \Sigma_{l_1}(l_1 \setminus \{0\}) \times \Sigma_{l_1} l_1(wB)$$
$$\to \Sigma_{l_1}(l_1 \setminus \{0\}) \times \Sigma_{l_1} B \times \Sigma_{l_1}(l_1 \setminus \{0\}) \times \Sigma_{l_1} l_1(wB)$$

satisfying the following properties:

(1) $h(r, 0) = (\psi_1(r), 0, \psi_2(r), 0)$ *where* $\psi: \Sigma_{l_1}(l_1 \setminus \{0\}) \to \Sigma_{l_1}(l_1 \setminus \{0\}) \times \Sigma_{l_1}(l_1 \setminus \{0\})$ *is the isomorphism sending odd coordinates to the first copy and even coordinates to the second copy and where* ψ_i *is projection of* ψ *into the ith copy of* $\Sigma_{l_1}(l_1 \setminus \{0\})$, $i = 1, 2$.

(2) *For each positive integer n there is a positive integer m such that*

$$h_4 \circ \underbrace{(h_3, h_4) \circ \cdots \circ (h_3, h_4)}_{m}(r, x) = 0$$

for all x with at most the first n coordinates nonzero. Here h_i is projection of h onto the ith coordinate.

(3) $|r|_{l_1} + |x|_w = |h_1(r, x)|_{l_1} + |h_2(r, x)|_B + |h_3(r, x)|_{l_1} + |h_4(r, x)|_w$ *where h_i is again projection of h onto the ith coordinate and where* $| \ |_{l_1}$, $| \ |_B$, *and* $| \ |_w$ *are the norms on the spaces* $\Sigma_{l_1} l_1$, $\Sigma_{l_1} B$ *and* $\Sigma_{l_1} l_1(wB)$ *respectively.*

Proof. $\Sigma_{l_1} l_1(wB) = l_1(wB)$. Therefore, Theorem 1 guarantees a homeomorphism $g': \Sigma_{l_1} l_1(wB) \to \Sigma_{l_1} B \times \Sigma_{l_1} l_1(wB)$. It was shown by Klee in [6] and [7] that the unit sphere of any infinite-dimensional Banach space B' is homeomorphic with any of its hyperplanes (subspace of deficiency one). Thus $B' = \mathbf{R} \times N$ and the unit sphere is homeomorphic with N. But any infinite-dimensional Banach space has an l_1 factor. (See [12, Remark 1].) Thus $N \cong \mathbf{R} \times N$, and the unit sphere of B' is homeomorphic with B'. Therefore, given $g'\colon \Sigma_{l_1} l_1(wB) \to \Sigma_{l_1} B \times \Sigma_{l_1} l_1(wB)$ there is a homeomorphism g^* from the unit sphere of $\Sigma_{l_1} l_1(wB)$ to the unit sphere of $\Sigma_{l_1} B \times \Sigma_{l_1} l_1(wB)$ under norm $|\ |_B + |\ |_w$. But then there is a radial homeomorphism

$$g\colon \Sigma_{l_1} l_1(wB) \to \Sigma_{l_1} B \times \Sigma_{l_1} l_1(wB),$$

$$x \mapsto \left(|x|_w g_1^*\left(\frac{x}{|x|_w}\right), |x|_w g_2^*\left(\frac{x}{|x|_w}\right)\right).$$

$$g^{-1}\colon \Sigma_{l_1} B \times \Sigma_{l_1} l_1(wB) \to \Sigma_{l_1} l_1(wB),$$

$$(y, z) \mapsto (|y|_B + |z|_{l_1}) g^{*-1}\left(\frac{y}{|y|_B + |z|_{l_1}}, \frac{z}{|y|_B + |z|_{l_1}}\right).$$

In both cases zero is sent to zero. Now g has the property that $g^{-1}(\Sigma_{l_1} B \times \{0\})$ is a radial subset of $l_1(wB)$.

Now, by a theorem of the author's [12], $B^\omega \cong \Sigma_{l_1} B$ for all infinite-dimensional Banach spaces. Therefore, by Lemma 2, there is a closed imbedding f of $\Sigma_{l_1} l_1(wB)$ into $\Sigma_{l_1} B \times \{0\} \subset \Sigma_{l_1} B \times \Sigma_{l_1} l_1(wB)$. Define the map

$$\begin{array}{ccc}
\Sigma_{l_1} l_1(wB) \times \{0\} & \xrightarrow{(g^{-1} \circ f) \times \mathrm{id}} & \Sigma_{l_1} l_1(wB) \times \{0\} \\
\cap & & \cap \\
\Sigma_{l_1} l_1(wB) \times \Sigma_{l_1} l_1(wB) & & \Sigma_{l_1} l_1(wB) \times \Sigma_{l_1} l_1(wB).
\end{array}$$

This is a closed imbedding, and therefore, by a theorem of Klee [8], there is a homeomorphism

$$G\colon \Sigma_{l_1} l_1(wB) \times \Sigma_{l_1} l_1(wB) \to \Sigma_{l_1} l_1(wB) \times \Sigma_{l_1} l_1(wB)$$

which extends this map.

Let $\phi\colon \Sigma_{l_1} l_1(wB) \to \Sigma_{l_1} l_1(wB) \times \Sigma_{l_1} l_1(wB)$ be the isomorphism sending odd coordinates to the first copy and even coordinates to the second. Define

$$h'\colon \Sigma_{l_1} l_1(wB) \to \Sigma_{l_1} B \times \Sigma_{l_1} l_1(wB)$$

by

$$h'(x) = (\mathrm{id} \times \phi^{-1}) \circ (\mathrm{id} \times \phi^{-1} \times \mathrm{id}) \circ (\mathrm{id} \times G \times \mathrm{id}) \circ (\mathrm{id} \times \phi^{-1} \times \mathrm{id} \times \mathrm{id})$$
$$\circ (g \times \mathrm{id} \times \mathrm{id} \times \mathrm{id}) \circ (\phi \times \phi) \circ \phi(x).$$

That is

$$
\begin{aligned}
\Sigma_{l_1} l_1(wB) &\xrightarrow{\ \phi\ } \Sigma_{l_1} l_1(wB) \times \Sigma_{l_1} l_1(wB) \\
&\xrightarrow{\ \phi\times\phi\ } \Sigma_{l_1} l_1(wB) \times \Sigma_{l_1} l_1(wB) \times \Sigma_{l_1} l_1(wB) \times \Sigma_{l_1} l_1(wB) \\
&\xrightarrow{\ g\times\mathrm{id}\times\mathrm{id}\times\mathrm{id}\ } \Sigma_{l_1} B \times \Sigma_{l_1} l_1(wB) \times \Sigma_{l_1} l_1(wB) \times \Sigma_{l_1} l_1(wB) \times \Sigma_{l_1} l_1(wB) \\
&\xrightarrow{\ \mathrm{id}\times\phi^{-1}\times\mathrm{id}\times\mathrm{id}\ } \Sigma_{l_1} B \times \Sigma_{l_1} l_1(wB) \times \Sigma_{l_1} l_1(wB) \times \Sigma_{l_1} l_1(wB) \\
&\xrightarrow{\ \mathrm{id}\times G\times\mathrm{id}\ } \Sigma_{l_1} B \times \Sigma_{l_1} l_1(wB) \times \Sigma_{l_1} l_1(wB) \times \Sigma_{l_1} i_1(wB) \\
&\xrightarrow{\ \mathrm{id}\times\phi^{-1}\times\mathrm{id}\ } \Sigma_{l_1} B \times \Sigma_{l_1} l_1(wB) \times \Sigma_{l_1} l_1(wB) \\
&\xrightarrow{\ \mathrm{id}\times\phi^{-1}\ } \Sigma_{l_1} B \times \Sigma_{l_1} l_1(wB).
\end{aligned}
$$

Since h' is a composition of homeomorphisms, h' is a homeomorphism.

Finally, define a new homeomorphism h by

$$h: \Sigma_{l_1}(l_1\setminus\{0\}) \times \Sigma_{l_1} l_1(wB) \to \Sigma_{l_1}(l_1\setminus\{0\}) \times \Sigma_{l_1} B \times \Sigma_{l_1}(l_1\setminus\{0\}) \times \Sigma_{l_1} l_1(wB),$$

$$p = (r, x) \mapsto \left(t_p\psi_1\left(\frac{r}{|r|_{l_1}}\right), t_p h_1'\left(\frac{x}{|r|_{l_1}}\right), t_p\psi_2\left(\frac{r}{|r|_{l_1}}\right), t_p h_2'\left(\frac{x}{|r|_{l_1}}\right)\right)$$

where

$$t_p = \frac{|r|_{l_1} + |x|_w}{1 + |h_1'(x/|r|_{l_1})|_B + |h_2'(x/|r|_{l_1})|_w}.$$

Now h is continuous since it is coordinate-wise continuous. (Note that $|r|_{l_1}$ cannot be zero.)

$$h^{-1}: \Sigma_{l_1}(l_1\setminus\{0\}) \times \Sigma_{l_1} B \times \Sigma_{l_1}(l_1\setminus\{0\}) \times \Sigma_{l_1} l_1(wB) \to \Sigma_{l_1}(l_1\setminus\{0\}) \times \Sigma_{l_1} l_1(wB)$$

$$
\begin{aligned}
q: (r, y, s, z) \mapsto \Bigg(& t_q\psi^{-1}\left(\frac{r}{|r|_{l_1} + |s|_{l_1}}, \frac{s}{|r|_{l_1} + |s|_{l_1}}\right), \\
& t_q h'^{-1}\left(\frac{y}{|r|_{l_1} + |s|_{l_1}}, \frac{z}{|r|_{l_1} + |s|_{l_1}}\right)\Bigg)
\end{aligned}
$$

where

$$t_q = \frac{|r|_{l_1} + |y|_B + |s|_{l_1} + |z|_w}{1 + |h'^{-1}(y/(|r|_{l_1} + |s|_{l_1}), z/(|r|_{l_1} + |s|_{l_1}))|_w}.$$

The reader may now show that h is the required homeomorphism. Remember that $g^{-1}(\Sigma_{l_1} B \times \{0\})$ is radial. Also, look at the diagram above. When $x \in \Sigma_{l_1} l_1(wB)$ and $\phi(x)$ has second coordinate zero, then $h_2' \circ h'(x)$ will be zero. This will give us condition (2). □

A closed set $K \subset B$ has *property* Z (is a Z-set) in B if for each nonempty, homotopically trivial, open set U in B it is true that $U \backslash K$ is nonempty and homotopically trivial.

We will need the following theorem concerning Z-sets:

Theorem 4. *Given a Banach space $B \cong B^\omega$, then a countable union of Z-sets $\bigcup_{i=1}^\infty K^i$ is negligible in B, i.e. $B \cong B \backslash \bigcup_{i=1}^\infty K^i$.*

Proof. This result is due to Chapman and Toruńczyk, independently. See [2] or [14]. □

Remark 2. $\Sigma_{l_1}(l_1 \backslash \{0\}) \cong \Sigma_{l_1} l_1 = l_1$. This is due to the fact that $K_n = \{\{x_i\} \in \Sigma_{l_1} l_1 | x_n = 0\}$ is a Z-set in $\Sigma_{l_1} l_1$ and $(\Sigma_{l_1} l_1) \backslash (\bigcup_{n=1}^\infty K_n) = \Sigma_{l_1}(l_1 \backslash \{0\})$. See Cutler [3, Theorem 1] for a proof that $(\Sigma_{l_1} l_1) \backslash (\bigcup_{n=1}^\infty K_n)$ is homeomorphic with $\Sigma_{l_1} l_1$. $\Sigma_{l_1}(\Sigma_{l_1}(l_1 \backslash \{0\})) \cong l_1$ by a similar proof.

3. Main results.

Lemma 5. *Any infinite-dimensional Banach space $B \cong B^\omega$ is homeomorphic to $l_1(wB)$.*

Proof. Let h be the homeomorphism guaranteed by Lemma 3,

$$h: \Sigma_{l_1}(l_1 \backslash \{0\}) \times \Sigma_{l_1} l_1(wB) \to \Sigma_{l_1}(l_1 \backslash \{0\}) \times \Sigma_{l_1} B \times \Sigma_{l_1}(l_1 \backslash \{0\}) \times \Sigma_{l_1} l_1(wB),$$

and let h_i be the projection of h onto the ith coordinate, $i = 1, 2, 3, 4$. Define

$$\Sigma_{l_1}^f(\Sigma_{l_1} B) = \{\{x_i\} \in \Sigma_{l_1}(\Sigma_{l_1} B) | x_i = 0 \text{ for almost all } i\},$$

and define

$$A = \{(r, x) \in \Sigma_{l_1}(l_1 \backslash \{0\}) \times \Sigma_{l_1} l_1(wB)|$$
$$h_4 \circ \underbrace{(h_3, h_4) \circ \cdots \circ (h_3, h_4)}_{n-1}(r, x) = 0 \text{ for some } n = 1, 2, \cdots\}$$

where

$$(h_3, h_4): \Sigma_{l_1}(l_1 \backslash \{0\}) \times \Sigma_{l_1} l_1(wB) \to \Sigma_{l_1}(l_1 \backslash \{0\}) \times \Sigma_{l_1} l_1(wB),$$

$$(r, x) \mapsto (h_3(r, x), h_4(r, x)).$$

We will let $(h_3, h_4)^n$ denote the n-fold composition of (h_3, h_4). (Note that A is a set of the form $\Sigma_{l_1}(l_1 \backslash \{0\}) \times N$, i.e. A does not depend on the first coordinate.) Then define

$$H: A \to \Sigma_{l_1}(\Sigma_{l_1}(l_1 \backslash \{0\})) \times \Sigma_{l_1^f}(\Sigma_{l_1} B),$$

$$(r, x) \mapsto (\{h_1 \circ (h_3, h_4)^{n-1}(r, x)\}_n, \{h_2 \circ (h_3, h_4)^{n-1}(r, x)\}_n).$$

H is into by condition (1) of Lemma 3. Given $(r, x) \in A$, there exists an n such that $h_4 \circ (h_3, h_4)^n(r, x) = 0$. Then let $s = h_3 \circ (h_3, h_4)^n(r, x)$. By condition (1), $h(s, 0) = (\psi_1(s), 0, \psi_2(s), 0)$. But ψ is an isomorphism, and therefore, $H(r, x)$ is summable. In fact, $|H(r, x)|_B = |(r, x)|_A$ where $|\ |_A$ is the norm on A and $|\ |_B$ is the norm on $\Sigma_{l_1}(\Sigma_{l_1}(l_1 \backslash \{0\})) \times \Sigma_{l_1^f}(\Sigma_{l_1} B)$.

Now define

$${}^1h = h = (h_1, h_2, h_3, h_4)$$

$${}^2h = (h_1, h_2, h \circ (h_3, h_4))$$

$$\vdots$$

$${}^nh = (h_1, h_2, h_1 \circ (h_3, h_4), h_2 \circ (h_3, h_4), \cdots,$$
$$h_1 \circ (h_3, h_4)^{n-2}, h_2 \circ (h_3, h_4)^{n-2}, h \circ (h_3, h_4)^{n-1}).$$

$${}^nh: \Sigma_{l_1}(l_1 \backslash \{0\}) \times \Sigma_{l_1} l_1(wB) \to \underbrace{[(\Sigma_{l_1}(l_1 \backslash \{0\}) \times \Sigma_{l_1} B) \times \cdots \times (\Sigma_{l_1}(l_1 \backslash \{0\}) \times \Sigma_{l_1} B)]}_{n-2}$$
$$\times \Sigma_{l_1}(l_1 \backslash \{0\}) \times \Sigma_{l_1} B \times \Sigma_{l_1}(l_1 \backslash \{0\}) \times \Sigma_{l_1} l_1(wB).$$

${}^nh$ is a homeomorphism for each n and ${}^nh$ is distance preserving from zero by condition 3 of Lemma 3.

To see that H is continuous, given $\{(r^j, x^j)\}_j$ converging to (r, x) in A and $\epsilon > 0$, pick N such that

$$\sum_{i=N+1}^{\infty} [|h_1 \circ (h_3, h_4)^{i-1}(r, x)| + |h_2 \circ (h_3, h_4)^{i-1}(r, x)|] < \epsilon/8.$$

Now since ${}^{N+1}h$ is a homeomorphism, given $\epsilon/8$ there exists a J such that $j > J$ implies

$$\sum_{i=1}^{N} |b_1 \circ (b_3, b_4)^{i-1}(r, x) - b_1 \circ (b_3, b_4)^{i-1}(r^j, x^j)|$$

$$+ \sum_{i=1}^{N} |b_2 \circ (b_3, b_4)^{i-1}(r, x) - b_2 \circ (b_3, b_4)^{i-1}(r^j, x^j)|$$

$$+ |b_3 \circ (b_3, b_4)^{N}(r, x) - b_3 \circ (b_3, b_4)^{N}(r^j, x^j)|$$

$$+ |b_4 \circ (b_3, b_4)^{N}(r, x) - b_4 \circ (b_3, b_4)^{N}(r^j, x^j)| < \frac{\epsilon}{8}.$$

In particular, the last two terms in the four term sum are less than $\epsilon/8$. Since H is distance preserving from zero, $|H(r, x) - H(r^j, x^j)| < \epsilon$. To see this, note that

$$|b_3 \circ (b_3, b_4)^{N}(r, x) - b_3 \circ (b_3, b_4)^{N}(r^j, x^j)| < \frac{\epsilon}{8}$$

says that $|b_3 \circ (b_3, b_4)^N(r^j, x^j)| < \epsilon/8 + \epsilon/8$ since

$$|b_3 \circ (b_3, b_4)^N(r, x)| + |b_4 \circ (b_3, b_4)^N(r, x)|$$

$$= \sum_{i=N+1}^{\infty} [|b_1 \circ (b_3, b_4)^{i-1}(r, x)| + |b_2 \circ (b_3, b_4)^{i-1}(r, x)|].$$

The same holds for $|b_4 \circ (b_3, b_4)^N(r^j, x^j)|$. H is clearly one-to-one and onto.

To see that H^{-1} is continuous, let $\{(r^j, x^j)\}_j$ converge to (r, x) in $\Sigma_{l_1}(\Sigma_{l_1}(l_1\backslash\{0\})) \times \Sigma_{l_1^f}(\Sigma_{l_1}B)$ and suppose we are given $\epsilon > 0$. Let N_0' be a positive integer such that $x_n = 0$ for $n \geq N_0'$ where x_n is the nth coordinate in $x = (x_i)$. Then pick $N_0 \geq N_0'$ so that $\Sigma_{N_0+1}^{\infty}(|r_i| + |x_i|) < 1$. Then, let $(s, 0)$ with $|s| < 1$ be the last two coordinates in ${}^{N_0}b(H^{-1}(r, x))$. Similarly, let (s^j, y^j) be the last two coordinates in ${}^{N_0}b(H^{-1}(r^j, x^j))$. Next, pick J such that $j \geq J$ implies $|s^j| < 1$. Now, ${}^{N_0}b$ is a homeomorphism. Therefore, given ${}^{N_0}b(H^{-1}(r, x))$ and $\epsilon > 0$ there is a $\delta > 0$ such that if a point is within δ of ${}^{N_0}b(H^{-1}(r, x))$ then its image under $({}^{N_0}b)^{-1}$ is within ϵ of $H^{-1}(r, x)$.

Next, pick $N > N_0$ so that $\Sigma_{N+1}^{\infty}|\psi_1 \circ \psi_2 \circ \cdots \circ \psi_2(s)| < \delta/16$. Then, let $\delta_0 = \delta/4N^2$. Given δ_0, pick $\eta \leq \delta_0$ such that $z \in \Sigma_{l_1}l_1(wB)$ and $|z - 0|_w < \eta$ implies

$$|b_1'(z) - 0|_B + |b_2'(z) - 0|_w < \delta_0.$$

Then pick $J_0' \geq J$ such that $j > J_0'$ implies

$$\frac{|y^j|}{|s^j|} < \eta, \ \frac{|b_4 \circ (b_3, b_4)^{n-1}(s^j, y^j)|}{|b_3 \circ (b_3, b_4)^{n-1}(s^j, y^j)|} < \eta, \quad \text{for } n = 1, 2, \cdots, N.$$

Then $j \geq J_0'$ (looking at the definition of b)

$$\left|\frac{(|s^j|+|y^j|)\psi_1(s^j/|s^j|)}{1+|b_1'(y^j/|s^j|)|+|b_2'(y^j/|s^j|)|}-\psi_1(s^j)\right|$$

$$=\left|\frac{(|s^j|+|y^j|)\psi_1(s^j/|s^j|)}{1+|b_1'(y^j/|s^j|)|+|b_2'(y^j/|s^j|)|}-|s^j|\psi_1(s^j/|s^j|)\right|$$

$$\leq\left|\frac{|s^j|+|y^j|-|s^j|-|s^j|(|b_1'(y^j/|s^j|)|+|b_2'(y^j/|s^j|)|)}{1+|b_1'(y^j/|s^j|)|+|b_2'(y^j/|s^j|)|}\right|$$

$$\leq\big||y^j|-|s^j|(|b_1'(y^j/|s^j|)|+|b_2'(y^j/|s^j|)|)\big|\leq\delta_0=\delta/4N^2.$$

Similarly

$$\left|\frac{(|s^j|+|y^j|)\psi_2(s^j/|s^j|)}{1+|b_1'(y^j/|s^j|)|+|b_2'(y^j/|s^j|)|}-\psi_2(s^j)\right|\leq\delta_0.$$

Also, let

$$p=\frac{(|s^j|+|y^j|)\psi_2(s^j/|s^j|)}{1+|b_1'(y^j/|s^j|)|+|b_2'(y^j/|s^j|)|}=b_3(s^j,y^j),$$

and let $q=b_4(s^j,y^j)$. Then, by the same argument as above

$$\left|\frac{(|p|+|q|)\psi_1(p/|p|)}{1+|b_1'(q/|p|)|+|b_2'(q/|p|)|}-\psi_1(p)\right|\leq\big||q|-|p|(|b_1'(q/|p|)|+|b_2'(q/|p|)|)\big|\leq\delta_0.$$

But $|\psi_1\circ\psi_2(s^j)-\psi_1(p)|\leq|\psi_2(s^j)-p|<\delta_0=\delta/4N^2$ and ψ is an isomorphism. Therefore

$$\left|\psi_1\circ\psi_2(s^j)-\frac{(|p|+|q|)\psi_1(p/|p|)}{1+|b_1'(q/|p|)|+|b_2'(q/|p|)|}\right|$$

$$\leq|\psi_1\circ\psi_2(s^j)-\psi_1(p)|+\left|\frac{(|p|+|q|)\psi_1(p/|p|)}{1+|b_1'(q/|p|)|+|b_2'(q/|p|)|}-\psi_1(p)\right|\leq 2\delta/4N^2.$$

By induction, the kth coordinate for $k\leq N$ satisfies

$$|\psi_1\circ\psi_2^{k-1}(s^j)-b_1\circ(b_3,b_4)^{k-1}(s^j,y^j)|<k\delta/4N^2.$$

Here ψ_2^{k-1} denotes the $(k-1)$-fold composition of ψ_2. But now pick $J_0\geq J_0'$ such that for $j>J_0$

(1) $|\psi_1 \circ \psi_2^{k-1}(s) - b_1 \circ (b_3, b_4)^{k-1}(s^j, y^j)| < \delta/4N$ for $k = 1, 2, \cdots, N$;
(2) $|y^j| < \delta/4$; and
(3) $\|(s^j, y^j)| - |(s, 0)\| < \delta/16$.

Then

$$\begin{aligned}
|s^j - s| + |y^j - 0| &= \sum_{i=1}^{N} |\psi_1 \circ \psi_2^{i-1}(s) - \psi_1 \circ \psi_2^{i-1}(s^j)| + |y^j| \\
&\quad + \sum_{i=N+1}^{\infty} |\psi_1 \circ \psi_2^{i-1}(s) - \psi_1 \circ \psi_2^{i-1}(s^j)| \\
&\leq \sum_{i=1}^{N} |\psi_1 \circ \psi_2^{i-1}(s) - b_1 \circ (b_3, b_4)^{i-1}(s^j, y^j)| \\
&\quad + \sum_{i=1}^{N} |b_1 \circ (b_3, b_4)^{i-1}(s^j, y^j) - \psi_1 \circ \psi_2^{i-1}(s^j)| \\
&\quad + \frac{\delta}{4} + \sum_{i=N+1}^{\infty} |\psi_1 \circ \psi_2^{i-1}(s)| + \sum_{i=N+1}^{\infty} |\psi_1 \circ \psi_2^{i-1}(s^j)| \\
&\leq N(\delta/4N) + N(\delta/4N) + \delta/4 + \delta/16 + (\delta/16 + \delta/16) \\
&= 15\delta/16.
\end{aligned}$$

Therefore, pick $J_1 \geq J_0$ such that $j \geq J_1$ implies

$$|\pi_{N_0} \circ [{}^{N_0}b(H^{-1}(r, x))] - \pi_{N_0} \circ [{}^{N_0}b(H^{-1}(r^j, x^j))]| < \delta/16,$$

where π_{N_0} is projection onto the first N_0 coordinates. Then $|H^{-1}(r, x) - H^{-1}(r^j, x^j)| < \epsilon$ whenever $j \geq J_1$. Thus H^{-1} is continuous and H is a homeomorphism.

By Remark 2, $\Sigma_{l_1}(l_1 \backslash \{0\}) \cong l_1$ and $\Sigma_{l_1}(\Sigma_{l_1}(l_1 \backslash \{0\})) \cong l_1$. Using the fact that $A = \Sigma_{l_1}(l_1 \backslash \{0\}) \times N$ for some $N \subset \Sigma_{l_1} l_1(wB)$, we may change H to a homeomorphism H',

$$H' : l_1 \times N \to l_1 \times \Sigma_{l_1}(\Sigma_{l_1} B).$$

H' may be extended to G_δ subsets of $l_1 \times \Sigma_{l_1} l_1(wB)$ and $l_1 \times \Sigma_{l_1}(\Sigma_{l_1} B)$ by a theorem of Lavrentiev (see Kuratowski [9, p. 429]). But these G_δ sets are dense in the respective spaces, and the complements are countable unions of Z-sets. They are Z-sets since we may leave the l_1 coordinates alone and use the fact that $l_1 \times \Sigma_{l_1^f} l_1(wB)$ and $l_1 \times \Sigma_{l_1^f}(\Sigma_{l_1} B)$ are contained in the respective G_δ's. Then given a closed set K in the complement of, say, $l_1 \times \Sigma_{l_1^f} l_1(wB)$ and a map $f\colon S^n \to \mathfrak{U}\backslash K$, cover the image of S^n under f, a compact set, by a finite number of convex open sets contained in $\mathfrak{U}\backslash K$. Then pick an M such that $\pi_{l_1}(f(S^n)) \times [\pi_M(\pi_w f(S^n)) \times \{0\}]$ is contained in the union of the open sets. Here π_M is projection onto the first M coordinates and π_w is projection onto $\Sigma_{l_1} l_1(wB)$. Now, straight line homotopy each point of $f(S^n)$ to its corresponding point with $\{0\}$ from $M+1$ on in the $\Sigma_{l_1} l_1(wB)$ factor. This gives an extension of f to a function $\bar{f}\colon E^{n+1} \to \mathfrak{U}\backslash K$. Hence K is a Z-set.

Finally, by Theorem 4, $l_1 \times \Sigma_{l_1} l_1(wB) \cong l_1 \times \Sigma_{l_1}(\Sigma_{l_1}(B))$. But $l_1 \times \Sigma_{l_1} l_1(wB) \cong l_1(wB)$ since $\Sigma_{l_1} l_1(wB) = l_1(wB)$ and $l_1(wB) \cong l_1 \times l_1(wB)$. Similarly $l_1 \times \Sigma_{l_1}(\Sigma_{l_1} B) \cong \Sigma_{l_1} B$. By a theorem of the author's, [12]. $\Sigma_{l_1} B \cong B^\omega$. Therefore, since $B \cong B^\omega$ by assumption, $B \cong l_1(wB)$. □

To extend this result to Fréchet spaces, we need the following obvious lemma:

Lemma 6. *Let $\{|\ |_i\}$ be a collection of pseudo-norms which determine the topology of the Fréchet space F. Then there is, for each i, a continuous linear surjection T_i of F onto the Banach space $F/|\ |_i = B_i$.*

Theorem 7. *Any infinite-dimensional Fréchet space $F \cong F^\omega$ is homeomorphic with $l_2(wF)$.*

Proof. By Lemma 6, there are continuous linear surjections $T_i\colon F \to B_i$. By a result in [1], this gives us that $F \cong B_i \times N_i$ where N_i is the kernel of T_i. Therefore, since $F \cong F^\omega$,

$$F \cong B_i \times N_i \cong (B_i \times N_i)^\omega \cong B_i^\omega \times (B_i \times N_i)^\omega \cong B_i^\omega \times F.$$

Therefore,

$$F^\omega \cong \prod_{i=1}^{\infty} (B_i^\omega \times F) \cong \left(\prod_{i=1}^{\infty} B_i^\omega\right) \times F.$$

By a theorem of the author's [12],

$$B_i^\omega \cong \Sigma_{l_1} B_i \quad \text{and} \quad \prod_{i=1}^{\infty} \Sigma_{l_1} B_i \cong \Sigma_{l_1}(\Sigma_{l_1} B_i)_i.$$

Therefore, $F^\omega \cong \Sigma_{l_1}(\Sigma_{l_1} B_i)_i \times F$. But $\Sigma_{l_1}(\Sigma_{l_1} B_i)_i$ is a Banach space with weight equal to the weight of F, and it is homeomorphic with its countable product. Therefore, $\Sigma_{l_1}(\Sigma_{l_1} B_i)_i \cong l_1(wF)$. But then $F \cong l_1(wF) \times F$, and Theorem 1 gives $l_1(wF) \cong l_1(wF) \times F$. Thus, $F \cong l_1(wF)$. But $l_1(wF)$ is homeomorphic with $l_2(wF)$. (See Mazur, [10].) Therefore, $F \cong l_2(wF)$. □

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